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The Choice of Numerical Boundary Conditions for Hyperbolic Systems

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The aim of this paper is the discussion of two fundamental problems for mixed initial boundary value problems with applications in fluid mechanics. First, different stability properties are discussed, which are of importance for long time integrations and steady state calculations. Second, a new numerical technique for problems with an artificial boundary is introduced.

1. INTRODUCTION

The general term *stability* for a difference approximation to a time-dependent problem can be given many different interpretations. For ordinary differential equations there are presently so many different kinds of stability defined, that the alphabet seems to be too short for the one-letter type labeling starting with A-stability. For partial differential equations there are fewer definitions. All the different stability definitions for ODEs can of course be applied to a PDE once the dscretization in space has been made. In fact, the procedure of semi-discretization followed by the use of a standard ODE solver for the resulting time-dependent system

$$\partial u/\partial t = Pu \tag{1}$$

has gained popularity. For this so-called "method of lines," the ODE-stability theory is frequently used. One must, however, be aware that system (1) depends on the step size h, and the number of equations is unbounded when we consider arbitrary small h.

When one is interested in the solution over large time intervals, or when a time-dependent method is used for obtaining a steady state solution, all methods which allow growing solutions are of course useless. On the other hand, a method which for a given fixed step size gives solutions which converge to a steady state solution might have very bad stability properties. Recently Yee *et al.* [8] defined P-stability for initial boundary value problems, such that stability holds in the sense of Gustafsson *et al.* [4, Definition 3.3] and furthermore such that (almost) no growing solutions are

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allowed. The last condition is made precise by requiring that the operator Q in the difference scheme written in one-step form

$$u^{n+1} = Qu^n \tag{2}$$

have no eigenvalues outside the unit circle.

In this paper we shall analyze these matters further for general hyperbolic initial boundary value problems. Sufficient conditions are given such that all the eigenvalues of Q are inside the unit circle.

If the equations are defined on a domain which is unbounded in space, the most common computational technique are based on the introduction of an artificial boundary. If the hyperbolic system has characteristics pointing into the computational domain, then extrapolation procedures do not work well, as was shown in [5]. The construction of stable and convenient conditions at artificial boundaries has been considered by many authors. Engquist and Majda [2] designed absorbing boundary conditions, Hedstrom [6] constructed a similar type, Rudy and Strikwerda [7] considered the steady state problem for the Navier–Stokes equations, and managed to speed up the convergence rate by introducing a parameter in the boundary condition. Bayliss and Turkel [1] derived a downstream boundary condition for the Euler equations using the asymptotic behaviour of the wave equation.

In Section 3 we shall introduce a new numerical technique which is based on the very natural requirement that the solution remain bounded on the infinite domain. It is applied to the down stream boundary problem in fluid dynamics for the Euler equations. The full report on this latter work will be presented in a joint work with Ferm [3].

2. Stability

It is well known that the numerical boundary conditions for a hyperbolic problem may introduce instabilities to a difference approximation which is stable for the Cauchy problem. One reason for this sensitivity is that the energy of the true solution is almost conserved, the only dissipation is created through the boundary. As an example, consider the simple problem

$$\begin{aligned} u_t &= u_x, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t, \\ u(1, t) &= 0, \qquad u(x, 0) = f(x). \end{aligned} \tag{3}$$

With the norm defined by

$$\|u\|^2 = \int_0^1 u(x, t)^2 \, dx$$

we get immediately

$$(d/dt)\,\|u\|^2 = -u(0,t)^2. \tag{4}$$

A semi-discrete approximation is

$$\begin{aligned} &\partial u_j/\partial t = D_0 u_j, \qquad j = 1, 2, \ldots, N-1, \\ &u_N(t) = 0, \\ &u_j(0) = f_j, \qquad j = 0, 1, \ldots, N, \end{aligned} \tag{5}$$

where D_0 is the usual centered difference operator. An extra boundary condition is required at $j = 0$, and we use zeroth-order extrapolation

$$u_0(t) = u_1(t). \tag{6}$$

With the norm defined by

$$\|u\|^2 = \sum_{j=1}^{N-1} u_j(t)^2\, h$$

a simple calculation shows

$$\frac{d}{dt}\|u\|^2 = -u_1(t)\,u_0(t) = -u_0(t)^2 \tag{7}$$

which is completely analogous to (4). The accuracy of (6) is too low, however, and we may consider first-order extrapolation

$$u_0(t) = 2u_1(t) - u_2(t). \tag{8}$$

With the norm defined as above, we get

$$\frac{d}{dt}\|u\|^2 = -u_1(t)^2 - u_1(t)(u_1(t) - u_2(t)).$$

The last term destroys our estimate, and the sign of the right-hand side is unknown. Extra boundary condition (8) has introduced the possibility of an increasing energy represented by the norm chosen. This does not mean that the difference scheme is unstable; it can be shown that with another choice of norm, we get the energy dissipation back. The example only serves as an illustration of the sensitivity of the approximation to the choice of boundary conditions. The situation becomes even worse for the fully discretized problem. For example, if a centered difference operator is also used in time, resulting in the leapfrog scheme, it is well known that if the operator P in (1) has eigenvalues in the left half plane, there is a growing mode in the solutions to (2). This again does not by itself necessarily mean that the scheme is unstable, but in this case it can be shown by the normal mode analysis that it really is.

Sometimes it is argued that a sufficient test would be to compute the eigenvalues μ of the matrix Q in (2) and make sure that there is none outside the unit circle and only simple ones on the unit circle. The following example shows the insufficiency of such a procedure.

The Lax–Wendroff scheme for the model equation $u_t = u_x$ is

$$u_j^{n+1} = (I + kD_0 + (k^2/2) D_+ D_-) u_j^n, \qquad j = 1, 2, \ldots, N-1,$$

where D_+ and D_- are the forward and backward difference operators respectively. Instead of specifying the value at $j = N$, we reverse the boundary conditions used above and obtain

$$u_0^n = 0, \qquad u_N^n = u_{N-1}^n.$$

This simulates an extrapolation procedure at a subsonic downstream boundary, and an overspecification at a subsonic inflow boundary, see [5]. The resulting matrix representation of the operator Q in (2) is

$$Q = \begin{bmatrix} 1-\lambda^2 & \frac{\lambda}{2}+\frac{\lambda^2}{2} & & & & \\ -\frac{\lambda}{2}+\frac{\lambda^2}{2} & \ddots & \ddots & & 0 & \\ & \ddots & & & & \\ & \frac{\lambda}{2}+\frac{\lambda^2}{2} & 1-\lambda^2 & \frac{\lambda}{2}+\frac{\lambda^2}{2} & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & -\frac{\lambda}{2}+\frac{\lambda^2}{2} & 1-\lambda^2 & \frac{\lambda}{2}+\frac{\lambda^2}{2} & \\ & & & -\frac{\lambda}{2}+\frac{\lambda^2}{2} & 1+\frac{\lambda}{2}-\frac{\lambda^2}{2} \end{bmatrix},$$

where $\lambda \leqslant 1$ is the ratio between the time step k and the space step h. The eigenvectors $(v_1, v_2, \ldots, v_{N-1})^T$ must satisfy the relations

$$(-1+\lambda)(\lambda/2) v_{j-1} + (1-\lambda^2) v_j + (1+\lambda)(\lambda/2) v_{j+1} = \mu v_j, \qquad j = 1, 2, \ldots, N-1,$$

$$v_0 = 0, \qquad v_N = v_{N-1}. \tag{9}$$

The first equation is an ordinary difference equation which has the solution

$$v_j = \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j, \qquad \kappa_1 \neq \kappa_2, \tag{10}$$

where κ_1, κ_2 are the roots of

$$(-1+\lambda)(\lambda/2)+(1-\lambda^2)\kappa+(1+\lambda)(\lambda/2)\kappa^2=\mu\kappa. \tag{11}$$

The boundary conditions imply

$$\sigma_1+\sigma_2=0, \qquad (\kappa_1^N-\kappa_1^{N-1})\sigma_1+(\kappa_2^N-\kappa_2^{N-1})\sigma_2=0. \tag{12}$$

An eigenvector and corresponding eigenvalue μ exist if and only if this system has a nontrivial solution, and the condition for this is

$$\kappa_1^N-\kappa_1^{N-1}=\kappa_2^N-\kappa_2^{N-1}. \tag{13}$$

A straightforward calculation using (11) shows that if N is odd, all the eigenvalues μ are strictly inside the unit circle. Therefore the solutions u^n will eventually die out as n goes to infinity, and there might be a temptation to consider the scheme as stable for odd N. If, however, an inhomogenous term is introduced simulating rounding errors

$$\tilde{u}_j^{n+1}=Q\tilde{u}_j^n+\varepsilon_j^n, \tag{14}$$

the scheme behaves very poorly. The finer a mesh that is used (of course keeping the mesh ratio λ), the worse the solutions becomes, as can be see in Fig. 1. The reason is that the scheme is unstable, which is easily shown by the normal mode analysis [5]. It is sufficient to study each boundary separately, and in order to analyze the effect of

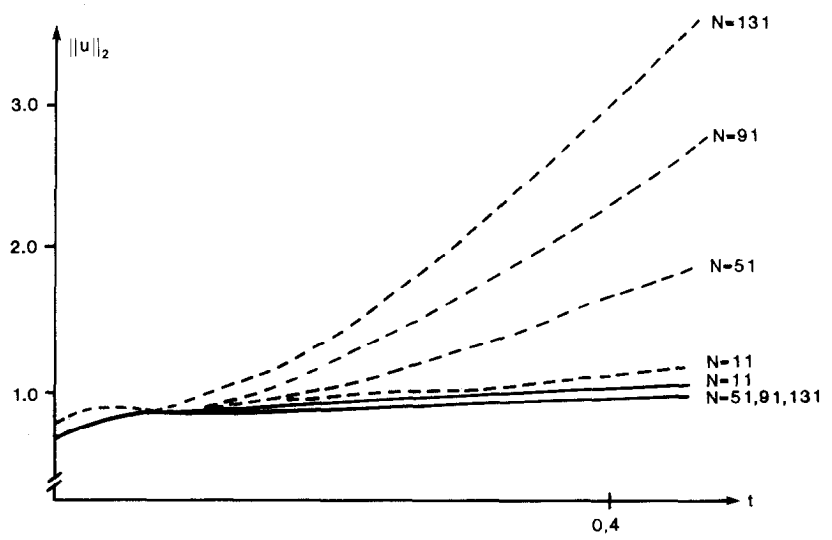

FIG. 1. Difference schemes with all eigenvalues $\mu(Q)$ inside the unit circle; (—) stable, (---) unstable.

the second condition in (9), we can also define the corresponding approximation for $u_t = -u_x$

$$u_j^{n+1} = (I - kD_0 + (k^2/2) D_+ D_-) u_j^n, \qquad j = 1, 2,...,$$
$$u_0^n = u_1^n, \qquad \sum_{j=0}^{\infty} (u_j^n)^2 h < \infty. \tag{15}$$

The normal mode analysis is analogous to the eigenvalue calculation above.

We look for nontrivial solutions to the resolvent equations

$$zv_j = (1 - kD_0 + (k^2/2) D_+ D_-) v_j, \qquad j = 1, 2,...,$$
$$v_0 = v_1, \qquad \sum_{j=0}^{\infty} v_j^2 h < \infty. \tag{16}$$

For $|z| > 1$, one can show that the last condition implies that there is only one mode in the solution, i.e., $v_j = \sigma\kappa^j$, where κ is the root of

$$z\kappa = (1 + \lambda)(\lambda/2) + (1 - \lambda^2)\kappa + (-1 + \lambda)(\lambda/2)\kappa^2$$

satisfying $|\kappa| < 1$ for $|z| > 1$. Obviously, a nontrivial solution exists if and only if $\kappa = 1$, which happens when $z = 1$. One says that there is a generalized eigenvalue at $z = 1$, and the approximation is unstable.

If, on the other hand, the boundary conditions are posed in the more normal way such that we have for the original problem

$$u_0^n = u_1^n, \qquad u_N^n = 0, \tag{17}$$

then the scheme is stable. The numerical experiments for this approximation with an inhomogenous term introduced are represented by the dashed line in Fig. 1. The norm is practically independent of the mesh size in this case. As discussed in the introduction, for long time integrations no growing modes can be allowed in the solution. Therefore it is natural to require that in addition to stability the operator Q in (2) have no eigenvalue outside the unit circle.

Since one may want to use the approximation for steady state calculations, it is actually desirable to require that all the eigenvalues of Q be *strictly* inside the unit circle. In that way convergence is guaranteed when the number of time steps tends to infinity. We say that the problem has only decreasing modes. The normal mode analysis and the stability theory in [4] have the advantage that a problem defined on a domain with two boundaries can be divided in two quarter space problems ($[0 \leqslant x] \times [0 \leqslant t]$ and $[x \leqslant 1] \times [0 \leqslant t]$) which are analyzed separately. We will investigate under what conditions any conclusions about decreasing modes for the two-boundary problem can be drawn from the analysis of the quarter space problems. We take the view that it is good enough if one can show that only decreasing modes

are present for N sufficiently large, provided the quarter space problems are stable. (From now on "stable" refers to [4, Definition 3.3].)

Let us consider the eigenvalues calculation for the Lax–Wendroff scheme in some more detail. The elements of the eigenvalues of the eigenvector v have form (10), and the different modes κ_1, κ_2 satisfy (11). For $|\mu| > 1$ the two roots are separated by the unit circle, and we define the roots such that $|\kappa_1| < 1$, $|\kappa_2| > 1$. Since the condition

$$\kappa_1 \kappa_2 = (\lambda - 1)/(\lambda + 1)$$

is always satisfied, there is a constant $\delta > 0$ independent of μ such that

$$|\kappa_1| \leqslant 1 - \delta, \qquad |\mu| \geqslant 1,$$
$$|\kappa_2| \geqslant 1, \qquad |\mu| \geqslant 1.$$

In fact one can prove that κ_2 approaches the unit circle only when μ approaches 1, and in this case $\kappa_2 = 1$. For stable boundary conditions (17) the coefficients σ_1, σ_2 in (10) satisfy

$$(1 - \kappa_1)\sigma_1 + (1 - \kappa^2)\sigma_2 = 0 \qquad \kappa_1^N \sigma_1 + \kappa_2^N \sigma^2 = 0 \tag{18}$$

and the condition for a nontrivial solution is

$$1 - \kappa_1 - r(N) = 0, \tag{19}$$

where

$$r(N) = (\kappa_1/\kappa_2)^N (1 - \kappa_2). \tag{20}$$

The term $1 - \kappa_1$ is well separated from zero and $r(N)$ vanishes when N tends to infinity. Therefore the result is

> The operator Q for the problem with two boundaries has all its eigenvalues strictly inside the unit circle if N is sufficiently large.

We will now generalize these results to larger classes of equations and approximations. Consider the general hyperbolic system with constant coefficients

$$u_t = A u_x. \tag{21}$$

Without restriction it can be assumed that the $(m \times m)$ matix A has diagonal form. The difference approximation is

$$Q_{-1} u^{n+1} = \sum_{v=0}^{s} Q_v u^{n-v}, \tag{22}$$

where Q_v are difference operators in space. Boundary conditions are specified at

$x = 0$ and $x = 1$. The symbols $\hat{Q}_v(\theta)$, $0 \leqslant |\theta| \leqslant \pi$ are obtained after Fourier transformation. It is assumed that the von Neumann condition is fulfilled, i.e., the equation

$$\left[z\hat{Q}_{-1}(\theta) - \sum_{v=0}^{s} z^{-v}\hat{Q}(\theta)\right] v = 0, \qquad v = (v^{(1)}, v^{(2)}, \ldots, v^{(m)})^{\mathrm{T}}, \tag{23}$$

has no nontrivial solution for $|z| > 1$. The resolvent equation for (22) is

$$zQ_{-1}v = \sum_{v=0}^{s} z^{-v}Q_v v. \tag{24}$$

This difference equation in space can be written in one-step form

$$w_{j+1} = Mw_j. \tag{25}$$

It is shown in [4] that M can be transformed to block diagonal form

$$T^{-1}(z)\, M(z)\, T(z) = \operatorname{diag}(L_1, L_2, N_1, N_2). \tag{26}$$

The properties of the blocks L_j, N_j when z approaches the unit circle are crucial for the stability theory. For a large class of approximations, class R, the following inequalities hold in the neighbourhood of any given point z_0 on the unit circle:

$$\begin{aligned} &L_1^* L_1 \leqslant (1-\delta) I, \qquad L_2^* L_2 \leqslant (1-\delta)(|z|-1) I, \\ &N_1^* N_1 \geqslant (1+\delta) I, \qquad N_2^* N_2 \geqslant (1+\delta)(|z|-1) I, \qquad \delta > 0. \end{aligned} \tag{27}$$

The eigenvalues of the different blocks are the roots κ_j to the characteristic equation derived for the Lax–Wendroff scheme above. In that case it was demonstrated that the block L_2 is empty near the whole unit circle, and that N_2 is empty except near $z^2 = 1$, where N_1 is empty. We shall prove the following general result for problems with two boundaries:

THEOREM. *Assume that both quarter space problems are stable and that inequalities* (27) *are fulfilled. If either* L_2 *or* N_2 *is empty, then there are only decreasing modes in the solutions to the two boundary problem for* N *sufficiently large.*

Proof. Without restriction it can be assumed that M has form (26). Assume first that L_2 is empty. Then w_j is partitioned correspondingly

$$w_j = (v_j, y_j^{(1)}, y_j^{(2)})^{\mathrm{T}}.$$

The stability assumptions imply that without restriction the boundary conditions can be written in the form

$$v_0 + D_1 y_0^{(1)} + D_2 y_0^{(2)} = 0, \qquad E_1 v_N + y_N^{(1)} = 0, \qquad E_2 v_N + y_N^{(2)} = 0$$

or equivalently

$$v_0 + D_1 y_0^{(1)} + D_2 y_0^{(2)} = 0,$$
$$-E_1 L_1^N (D_1 y_0^{(1)} + D_2 y_0^{(2)}) + N_1^N y_0^{(1)} = 0,$$
$$-E_2 L_1^N (D_1 y_0^{(1)} + D_2 y_0^{(2)}) + N_2^N y_0^{(2)} = 0.$$

Hence, the condition for a nontrivial solution is that the matrix

$$-E_2 L_1^N [D_1 (I - N_1^{-N} E_1 L_1^N D_1)^{-1} E_1 L_1^N D_2 + D_2] + N_2^N$$

be singular. Obviously, inequalities (27) makes this impossible for N sufficiently large. The case that N_2 is empty is treated completely analogously, and the theorem is proved.

It was shown in [4] that inequalities (27) are fulfilled for dissipative approximations.

The requirement of an empty L_2 or N_2 is normally fullfilled for systems (2) with all the eigenvalues of A having the same sign. This was proved for the Lax–Wendroff scheme above since (22) is a set of scalar equations. If the requirement is not fulfilled, the situation is still not too bad for dissipative approximations or "almost" dissipative approximations. The reason is that only those points where L_2 and N_2 are not empty require an investigation. We illustrate by an example. Consider the problem

$$w_t = A w_x, \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad w = \begin{bmatrix} u \\ v \end{bmatrix},$$
$$v(0, t) = 0, \qquad u(1, t) = v(1, t)$$

and the Lax–Wendroff approximation

$$w^{n+1} = (I + kAD_0 + (k^2/2) A^2 D_+ D_-) w^n$$

with the boundary conditions

$$v_0^n = 0, \qquad u_N^n = v_N^n, \qquad v_N^n = 2v_{N-1}^n - v_{N-2}^n, \qquad u_0^n = 2u_1^n - u_2^n.$$

Both quarter space problems are easily shown to be stable.

Using the same notation for the variables in the resolvent equation we have

$$u_j = \sigma_1 \kappa_1^j + \sigma_2 \kappa_2^j, \qquad v_j = \tau_1 \mu_1^j + \tau_2 \mu_2^j,$$

where κ_1, κ_2 and μ_1, μ_2 are roots to

$$z\kappa = \kappa + (\lambda/2)(\kappa^2 - 1) + (\lambda^2/2)(\kappa - 1)^2$$

and

$$z\mu = \mu - (\lambda/2)(\mu^2 - 1) + (\lambda^2/2)(\mu - 1)^2,$$

respectively.

The only crucial point where the conditions in the theorem are not fulfilled is $z = 1$, where $|\kappa_1| \leqslant 1 - \delta$, $\kappa_2 = 1$, $\mu_1 = 1$, $\mu_2 \geqslant 1 + \delta$. (The identification with the general notation in (26) is obtained through the relations $L_2 = \kappa_1$, $L_2 = \mu_1$, $N_1 = \mu_2$, $N_2 = \kappa_2$.) The equation for v, however, is independent of u and from the scalar result we know already that there is no eigenvalue z with $|z| \geqslant 1$ for N sufficiently large. The boundary conditions for u imply

$$\kappa_1^N \sigma_1 + \kappa_2^K \sigma_2 = \mu_1^N \tau_1 + \mu_2^N \tau_2,$$
$$(\kappa_1 - 1)^2 \sigma_1 + (\kappa_2 - 1)^2 \sigma_2 = 0.$$

Since $\tau_1 = \tau_2 = 0$ for $|z| \geqslant 1$, the scalar result for u applies, and there are obviously only decreasing solutions for N sufficiently large.

Let us go back to the original model example $u_t = u_x$ and study the backward Euler approximation

$$(I - kD_0)\, u_n^{n+1} = u_j^n, \qquad j = 1, 2, \ldots, N - 1,$$
$$u_0^n = 2u_1^n - u_2^n, \qquad u_N^n = 0.$$

The characteristic equation is

$$z(\kappa - (\lambda/2)(\kappa^2 - 1)) = \kappa. \tag{28}$$

For $\kappa = e^{i\theta}$ we have

$$z = 1/(1 - \lambda i \sin\theta),$$

and z hits the unit circle at $z = 1$ not only for $\theta = 0$ but also for $\theta = \pi$. For $z = 1$ the two roots to (28) are $\kappa_1 = -1$, $\kappa_2 = 1$, which shows that L_2 and N_2 in (26) are nonempty. This point, however, is easily investigated. The general solution to the resolvent equation for $z = 1$ is $v_j = \sigma_1(-1)^j + \sigma_2$ with σ_1, σ_2 satisfying

$$4\sigma_1 = 0, \qquad (-1)^N \sigma_1 + \sigma_2 = 0.$$

Obviously there is no nontrivial solution to this system, and therefore only decreasing solutions exist for N sufficiently large.

We end this section by making a comment about problems with nonconstant coefficients. The analysis above gives no information about the behaviour of the solutions to problems with variable coefficients. In fact, it must be expected that in such cases there exist growing solutions even if the analysis above shows decreasing solutions

for constant coefficients. We would like to emphasize that this is in general the correct behaviour. Consider the nonlinear model problem

$$u_t = uu_x, \qquad u(1, t) = 0, \qquad u(x, 0) = f(x)$$

with positive solutions u. The norm $\|u\|^2 = \int_0^1 u(x, t)^2\, dx$ is decreasing since

$$\frac{d}{dt}\|u\|^2 = -\frac{2}{3}u(0, t)^3.$$

The linearized problem is

$$u_t = a(x, t)\, u_x, \qquad a(x, t) > 0,$$
$$u(1, t) = 0, \qquad u(x, 0) = f(x),$$

and for the same norm we obtain

$$\frac{d}{dt}\|u\|^2 = \int_0^1 2auu_x\, dx = -a(0, t)\, u(0, t)^2 - \int_0^1 a_x^2 u^2\, dx.$$

Obviously there may be growing solutions to this problem. Therefore, if the full nonlinear problem cannot be analyzed, it makes sense to "jump over" the variable coefficient case and analyze the problem with constant coefficients.

3. Unbounded Domains

We shall consider an ideal fluid in a channel according to Fig. 2. The Euler equations are

$$w_t + A(w)\, w_x + B(w)\, w_y = 0, \tag{29}$$

where with the usual notation

$$w = \begin{bmatrix} \rho \\ u \\ v \end{bmatrix}, \qquad A(w) = \begin{bmatrix} u & \rho & 0 \\ c^2/\rho & u & 0 \\ 0 & 0 & u \end{bmatrix}, \qquad B(w) = \begin{bmatrix} v & 0 & \rho \\ 0 & v & 0 \\ c^2/\rho & 0 & v \end{bmatrix}.$$

The flow is assumed to be subsonic, but no data are available at D_β.

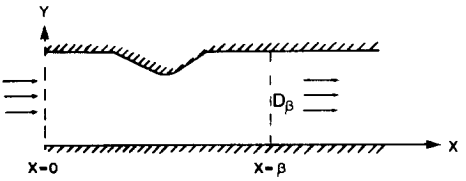

Figure 2

Seeking the steady state solution, we shall construct the downstream conditions using the condition

$$\sup_{x,y,t} |w| < \infty, \qquad |w|^2 = \rho^2 + u^2 + v^2. \tag{30}$$

The solution still contains an undetermined constant, therefore we add a condition on the massflow:

$$\int_0^1 \rho u \, dy = m. \tag{31}$$

Since m does not depend on x, it can be measured at the inflow boundary. The question is how these conditions can be converted into something usable in a computational procedure.

For the derivation it is assumed that the matrices A and B in (29) are constant, and since $v = 0$ at the boundary, we assume a zero diagonal in B; ρ and u are expanded in cosine series in the y direction; v is expanded in a sine series:

$$\rho(x, y) = \sum_{\omega=0}^{\infty}{}' \hat{\rho}_\omega(x) \cos \pi\omega y,$$

$$u(x, y) = \sum_{\omega=0}^{\infty}{}' \hat{u}_\omega(x) \cos \pi\omega y,$$

$$v(x, y) = \sum_{\omega=1}^{\infty}{}' \hat{v}(x) \sin \pi\omega y.$$

Introduction of these expansions into the steady state system gives the transformed equation

$$\bar{A} \frac{\partial \hat{w}_\omega}{\partial x} + \pi\omega \bar{B} \hat{w}_\omega = 0, \qquad w_\omega = \begin{bmatrix} \hat{\rho}_\omega \\ \hat{u}_\omega \\ \hat{v}_\omega \end{bmatrix}. \tag{32}$$

The solution has the form

$$\hat{w}_\omega = \sum_{j=1}^{3}{}' \alpha_j q_j e^{\omega\pi\lambda_j(x-\beta)}, \tag{33}$$

where the scalars λ_j and the vectors q_j satisfy the eigenvalue problem

$$(\bar{A}\lambda_j + \bar{B})\, q_j = 0. \tag{34}$$

Here λ_2 is positive, and condition (30) therefore implies $\alpha_2 = 0$ for $\omega \neq 0$. A straightforward calculation gives the final form of the solution

$$\begin{bmatrix} \rho_\omega \\ u_\omega \\ v_\omega \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -\bar{u}\bar{\rho}/\bar{c} \\ \bar{c} \\ s \end{bmatrix} e^{-(\omega\pi\bar{c}/s)(x-\beta)}, \qquad \omega = 1, 2, \ldots. \tag{35}$$

The quantities $\bar{u}$, $\bar{\rho}$ represent the coefficients in $\bar{A}$, $\bar{B}$; and s is defined by $s = \sqrt{\bar{c}^2 - \bar{u}^2}$. Equation (35) immediately gives the desired conditions to be used at $x = \beta$

$$\hat{\rho}_\omega = -(\bar{u}\bar{\rho}/\bar{c}s)\,\hat{v}_\omega, \qquad \omega = 1, 2, \ldots. \tag{36}$$

The remaining condition for $\hat{\rho}_0$ is obtained from (31).

It can be shown by the energy method that the system

$$\frac{\partial \hat{w}_\omega}{\partial t} + \bar{A}\frac{\partial \hat{w}_\omega}{\partial x} + \pi w \bar{B}\hat{w}_\omega = 0, \qquad \omega = 0, 1, \ldots, \tag{37}$$

is well posed with the boundary conditions derived above. Therefore methods for time-dependent problems can be used for computing the steady state solution.

In our numerical experiments we have used Newton's method for a difference approximation using centered differences in both directions. For comparison two

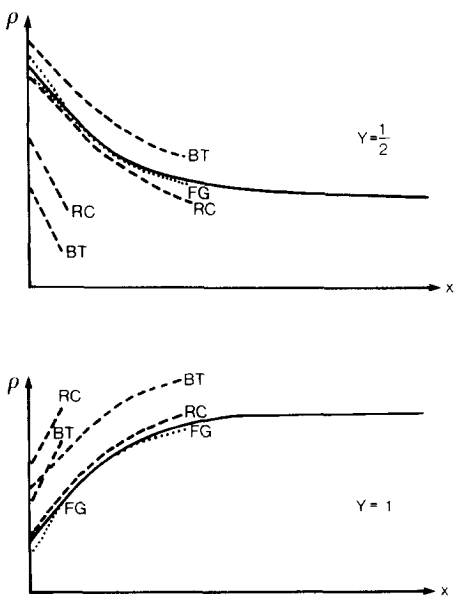

FIG. 3. Values of ρ as a function of x.

other boundary procedures have been used. In the first the conditions are based on the commonly used assumption that the solution has flattened out at $x = \beta$. If $w_t = w_x = 0$ in (29), it follows that $\rho_y = 0$. With the discretized version of this equation as the downstream boundary condition (everything else the same as above) a method is defined which is denoted by RC in Fig. 3.

The second alternative used for comparison is the Bayliss–Turkel method mentioned in the introduction, which is denoted by BT in the figure. It should be noted that this method can be modified such that it takes the solid walls into consideration, and in this way it would give better solutions [9].

The boundary D_β was placed at $x = \frac{5}{14}$ and $x = \frac{1}{14}$. The figure shows the remarkably good agreement between the correct solution and the one obtained with conditions (36) (denoted by FG).

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